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Proving the Rearrangeability of Connecting Networks by Group Calculations

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Concepts and calculations from group theory have led to a new way of demonstrating rearrangeability of networks made of stages of square switches and to new factorizations of symmetric groups of composite degree.

I. INTRODUCTION AND BACKGROUND

Telephone connecting networks usually consist of stages of switching that alternate with fixed cross-connect fields; in effect, these two kinds of units are used to build up desired connection patterns out of simpler permutations by *composition* (see Fig. 1). Since permutations form a group under composition, the notions of group theory have become relevant to the study of connecting networks. They are particularly useful for looking at desired combinatorial properties such as rearrangeability, which is the capacity to realize any permutation. This is true because, in the group-theoretic setting, the original Slepian-Duguid rearrangeability theorem[1] provides the possibility of factoring a symmetric group into a product of subgroups, or of double cosets of subgroups generated by stages.

Here we extend a natural notion of "switch permutation" implicit in Duguid's proof to general networks with nr inlets and as many outlets. For such networks μ and ν, we establish a group-theoretic condition on the sets $D(\mu)$ and $D(\nu)$ of switch permutations realized by μ and ν, respectively, under which the larger network obtained by cascading μ and ν alternately between three stages of r $n \times n$ switches is rearrangeable. This result corresponds to factorization of the symmetric group of degree nr into a product of subgroups with the sets $P(\mu)$ and $P(\nu)$ of permutations realized by μ and ν, respectively. The condition given is verified in the examples in Section V by carrying out group multiplications.

It is conceptually useful to regard a connecting network as a quadruple $\nu = (G, I, \Omega, S)$, where G is a graph depicting structure and, in particular, indicates between which terminals (nodes) there is a switch

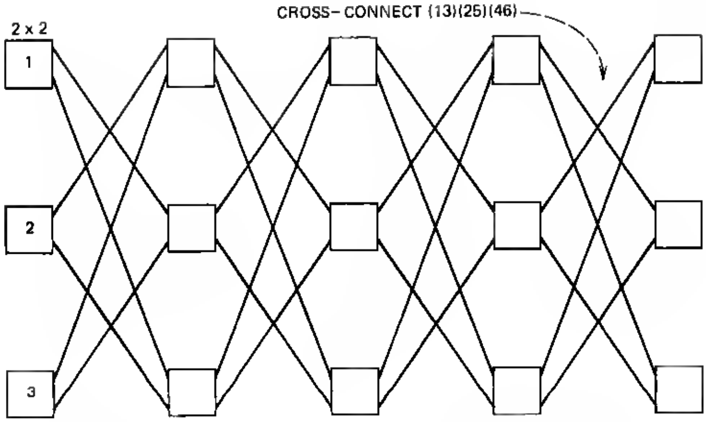


Fig. 1—Switching network with incomplete access between stages.

(edge); I and Ω are respectively the set of inlets (terminals) and the set of outlets, and S is the set of states deemed physically meaningful, that is, the set of allowed ways of closing switches so as to connect I to Ω hy paths through G. We shall assume ν to he two-sided: $I \cap \Omega = \phi$ and $|I| = |\Omega| = nr$, where n and r are integers $\geqq 2$. The set A of assignments is the set of correspondences of subsets of I into Ω, each correspondence heing interpreted as a particular way that terminals could ask to he connected together in pairs. Of course, there may or may not he a state in S realizing such a desired assignment. In any case, there is a natural map $\gamma: S \rightarrow A$ such that $\gamma(x)$ is the assignment realized by state x; in effect, $\gamma(x)$ tells us who is talking to whom when the network is in state x.

To put our questions into their natural group-theoretic setting, we shall identify hoth I and Ω with the integers $\{1, 2, \cdots, nr\}$, and the set of *maximal* assignments (everyhody wanting to talk to somehody) with S_{nr}, where

$$S_k = \{k - \text{permutations}\} = \text{symmetric group of degree } k.$$

The set $P(\nu)$ of maximal assignments or permutations realized by ν is then expressihle as

$$P(\nu) = \gamma(S) \cap S_{nr}.$$

A connecting network is called *rearrangeable* iff for every assignment $a \in A$ there is a state $x \in S$ such that x realizes a, i.e., $\gamma(x) = a$. Thus, the hasic problem of the rearrangeahility of ν can he cast in the following equivalent questions: When can every assignment he realized? When is $\gamma(S) = A$? Under our assumptions, these questions take the

form: When can the symmetric group on $\{1, \cdots, nr\}$ be realized? When is $P(\nu) = S_{nr}$?

The latter, group-theoretic form of the question begins to assume interest and importance when we note that many of the usual ways of constructing networks from stages of square switches correspond to *factoring* S_{nr} into factors that are subgroups. How this happens is explained next.

II. FACTORING S_{nr}

If X and Y are sets of group elements (complexes, in the old terminology) then XY is the set of products xy with $x \in X$ and $y \in Y$. We drop the notation I for the set of inlets, and use it henceforth for the identity permutation. Also, it is convenient to use exponent notation both for *products of complexes* with themselves, as X^2 for XX, and for the *direct product* of a group with itself some number of times. Thus, we establish the convention that if X is a complex, X^2 is XX as defined above; but if X is a group, then X^k means the k-fold direct product of X with itself.

It is readily seen, and has been pointed out before,[2] that a stage of square switches realizes an imprimitive subgroup of permutations. For example, the column of r $n \times n$ switches shown in the top half of Fig. 2 realizes the (imprimitive) subgroup that permutes the first n inlets among themselves, the second n among *themselves*, etc., up to the last n among each other. This subgroup is isomorphic to the direct product of S_n with itself r times, that is to $(S_n)^r$, and will be denoted by the same notation. In short, if ν is a stage of r $n \times n$ switches, then $P(\nu) = (S_n)^r$.

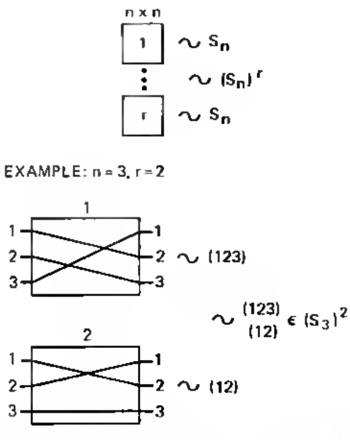


Fig. 2—Direct product group interpretation of a stage of square switches.

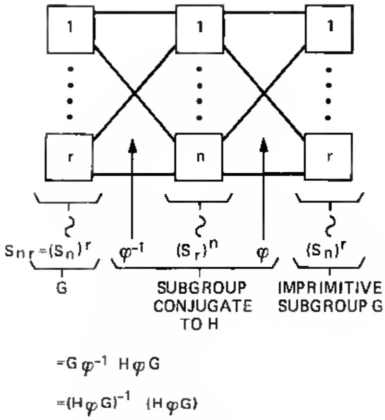


Fig. 3—Manner in which the three-stage network factors group S_{nr}, with $H = (S_r)^n$

Passing now to the three-stage network depicted in Fig. 3, we recall that by the classical result[3] of Slepian and Duguid, it is a rearrangeable network. We denote by φ the permutation corresponding to the standard cross-connect field between stages that defines a frame, namely,

$$\varphi\colon j \to 1 + [(j-1)/n] + r((j-1) \bmod n) \quad j = 1, \cdots, nr,$$

and we see that in Fig. 3 the middle and right stages have φ between them. (An alternative description of φ is that it takes the jth outlet on switch i into the ith outlet of switch j, for $j = 1, \cdots, n$ and $i = 1, \cdots, r$.) The original rearrangeability theorem can now be stated as a factorization, as follows (Fig. 3):

Classical Theorem (Slepian and Duguid): The symmetric three-stage network of square switches, in which switches on adjacent stages are connected by exactly one link, is rearrangeable and corresponds to a factorization

$$S_{nr} = (S_n)^r \varphi^{-1} (S_r)^n \varphi (S_n)^r. \tag{1}$$

The three middle factors above define a conjugate subgroup, so we have factored S_{nr} into a product of three subgroups. The remaining sections of this paper are devoted to finding alternative factorizations of S_{nr} that are associated with rearrangeable networks. We prove a factorization like (1) but with φ replaced by $P(\nu)$ for suitable ν, and then describe some applications.

III. SWITCH PERMUTATIONS

Now the essence of Duguid's proof of Slepian's result from Hall's theorem is contained in what we shall call a *switch-permutation*: he

decomposes any nr-permutation into a union of n submaps, each of which, because it corresponds basically to permuting outer switches, can be realized on a single middle switch. This idea is made precise as follows: define the function

$$\mathrm{sw}\colon \{1, \cdots, nr\} \to \{1, \cdots, r\}$$

by

$$\begin{aligned} \mathrm{sw}_i &= \text{the switch (inlet or outlet) } i \text{ is on in a stage of } r\ n \times n \text{ switches} \\ &= \text{the } k(1 \leqq k \leqq r) \text{ such that } nk - n + 1 \leqq i \leqq nk. \end{aligned}$$

Let π be a permutation of S_{nr}. A *Hall decomposition* of π is a partition $\pi = \bigcup_{l=1}^{n} p_l$ of π into n submaps p_l such that for $l = 1, \cdots, n$, the set

$$q_l = \{(\mathrm{sw}_i, \mathrm{sw}_j)\colon (i, j) \in p_l\}$$

is an r-permutation, i.e., $q_l \in S_r$. The intuitive meaning of this property of the p_l is that each one maps exactly one inlet from each consecutive set of n onto outlets that are on distinct consecutive sets of n outlets. Hall's theorem on distinct representatives of subsets implies:

Fact: Every $\pi \in S_{nr}$ has a Hall decomposition.

We can now define the switch-permutations generated by a network ν as follows: an element

$$\begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \in (S_r)^n$$

is a switch permutation generated by ν iff there exists $\pi \in P(\nu)$ with a Hall decomposition $\pi = \bigcup_{l=1}^{n} p_l$ such that

$$q_l = \{(\mathrm{sw}_i, \mathrm{sw}_j)\colon (i, j) \in p_l\}. \tag{2}$$

Remark 1: If $\begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}$ is a switch permutation generated by ν, then so is

$$\begin{pmatrix} q_{\pi(1)} \\ \vdots \\ q_{\pi(n)} \end{pmatrix}, \text{ for any } \pi \in S_n.$$

Intuitively, the q_l associated by (2) with the p_l of a Hall decomposition are just the settings of the successive middle switches that come out of Duguid's rearrangeability argument. The remark above is a reflection of the fact that submaps p_l of the decomposition can be assigned to the middle switches in an arbitrary way.

IV. FACTORIZATION

We let $D(\nu)$ he the set of switch permutations generated hy ν. The new factorization-rearrangeahility result we prove is as follows:

Theorem 1: *If μ and ν are networks with nr inlets and nr outlets, such that*

$$(S_r)^n \subseteq D(\nu)D(\mu),$$

then the network (Fig. 4) obtained by cascading ν and μ alternately between three stages of r $n \times n$ switches is rearrangeable, and corresponds to a factorization

$$S_{nr} = (S_n)^r P(\nu)(S_n)^r P(\mu)(S_n)^r.$$

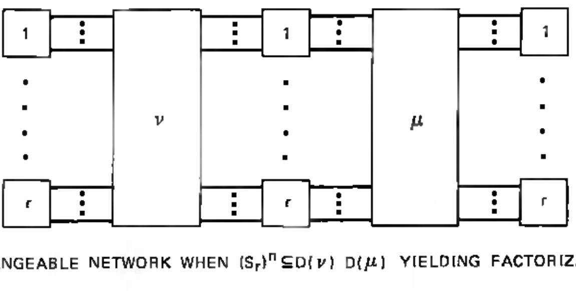
Proof: Take $\pi \in S_{nr}$ to he realized. It has a Hall decomposition $\pi = \bigcup_{l=1}^{n} p_l$ inducing a switch permutation

$$\begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \in (S_r)^n \subseteq D(\nu)D(\mu)$$

via $q_l = \{(\mathrm{sw}_i, \ \mathrm{sw}_j) \colon \ (i, \ j) \in p_l\}$ as hefore. Thus, for each $l = 1, \cdots, n$ there exist a_l and b_l each in S_r such that $q_l = b_l a_l$, with

$$\alpha = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in D(\mu) \text{ and } \beta = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in D(\nu).$$

The desired permutation can now he ohtained by setting μ and ν to generate switch permutations α and β, respectively. For $(i, j) \in p_l$ we look at how sw_i and sw_j are connected to the middle stage and claim that they are connected to the same middle-stage switch! This is hecause μ connects sw_i to $a_l(\mathrm{sw}_i)$, and ν connects sw_j to $b_l^{-1}(\mathrm{sw}_j)$.



REARRANGEABLE NETWORK WHEN $(S_r)^n \subseteq D(\nu)\ D(\mu)$ YIELDING FACTORIZATION

$S_{nr} = (S_n)^r\ P(\nu)\ (S_n)^r\ P(\mu)\ (S_n)^r$

Fig. 4—Rearrangeable network when $(S_r)^n \subseteq D(\nu)\ D(\mu)$, yielding factorization $S_{nr} = (S_n)^r P(\nu)(S_n)^r P(\mu)(S_n)^r$.

Since, by construction,

$$\begin{aligned} \mathrm{sw}_j &= q_l(\mathrm{sw}_i) \\ &= b_l[a_l(\mathrm{sw}_i)], \end{aligned}$$

we bave $a_l(\mathrm{sw}_i) = b_l^{-1}(\mathrm{sw}_j)$. It remains to route i to $a(\mathrm{sw}_i)$, to route j to $b_l^{-1}(\mathrm{sw}_j)$ and to complete the connection in the middle switch. This recipe works for all pairs $(i,\ j) \in \pi$, and tbe theorem is proved.

We next note that the hypothesis $(S_r)^n \subseteq D(\nu)\ \ D(\mu)$ of the theorem can be replaced by a stronger, more complicated condition that is less work to verify by calculation.

Remark 2: If M, N are subsets of $D(\mu)$, $D(\nu)$, respectively, such that for any $q_1, \cdots, q_n \in S_r$ tbere is some $\varphi \in S_n$ sucb that

$$\begin{pmatrix} q_{\varphi(1)} \\ \vdots \\ q_{\varphi(n)} \end{pmatrix} \in NM, \tag{3}$$

then $(S_r)^n \subseteq D(\nu)D(\mu)$. For if (3) holds, then tbere are a_l, b_l in M, N, respectively, and, hence, in $D(\nu)$ such that $q_{\varphi(l)} = b_1 a_l$, i.e., $q_l = b_{\varphi^{-1}(l)} a_{\varphi^{-1}(l)}$. But

$$\begin{pmatrix} a_{\varphi^{-1}(1)} \\ \vdots \\ a_{\varphi^{-1}(n)} \end{pmatrix} \in D(\mu), \quad \begin{pmatrix} b_{\varphi^{-1}(1)} \\ \vdots \\ b_{\varphi^{-1}(n)} \end{pmatrix} \in D(\nu)$$

by the remark following the definition of switch permutation. Hence,

$$\begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \in D(\nu)D(\mu).$$

V. EXAMPLES

Figure 5 and Tables I through III illustrate an application to tbe network of Fig. 1 to prove it rearrangeable. Here $\mu = \nu$, the network ν being just a stage of three 2×2 switcbes preceded and followed by tbe permutation (13) (25) (46) induced by the cross-connect field tbat links successive stages. Figure 5 illustrates two of the switch permutations generated by a copy of ν; the three stages shown in Fig. 5 are either tbe first three or the last three stages of the network of Fig. 1. Table I gives all eight possibilities; these form sets M, N (with $M = N$) of tbe form described in Remark 2, as can be verified from the product table, Table III, using the multiplication table for S_3 given in Table II. The entries of tbe product table that are shown form a subset C

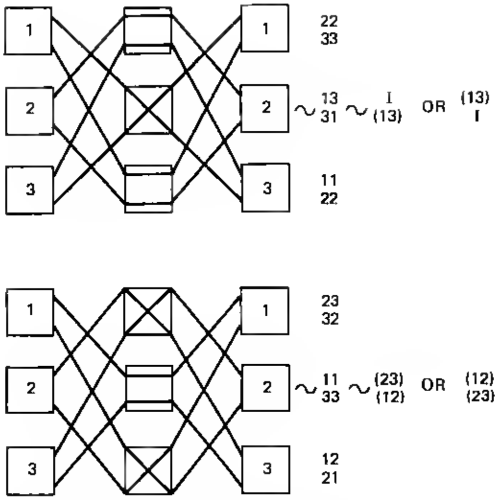


Fig. 5—Switch permutations generated by states of the middle stage.

of M such that either

$$\binom{a}{b} \quad \text{or} \quad \binom{b}{a}$$

belongs to C for any choice of a and b in S_3; thus, property (3) of Remark 2 holds, and Theorem 1 is applicable.

Tables IV and V show the same kind of calculation for the network with a cyclic cross-connect field (Fig. 6) that induces the permutation (5432). Tables VI and VII show the same results for the network (Fig. 7) based on (23) (45). Asterisks in Table VII define a subset

Table I — Direct product elements corresponding to switch settings for cross-connect (13) (25) (46) used in Fig. 1

sw #								
1st	22 33	23 32	22 33	22 33	23 32	23 32	22 33	23 32
2nd	11 33	11 33	13 31	11 33	13 31	11 33	13 31	13 31
3rd	11 22	11 22	11 22	12 21	11 22	12 21	12 21	12 21
Elements of K	I I	(23) I	I (13)	(12) I	(13) (23)	(23) (12)	(13) (12)	(123) (132)
	1	2	3	4	5	6	7	8

Table II — Multiplication table for S_3

		1st Operator					
		I	(12)	(13)	(23)	(123)	(132)
2nd Operator	I	I	(12)	(13)	(23)	(123)	(132)
	(12)	(12)	I	(132)	(123)	(23)	(13)
	(13)	(13)	(123)	I	(132)	(12)	(23)
	(23)	(23)	(132)	(123)	I	(13)	(12)
	(123)	(123)	(13)	(23)	(12)	(132)	I
	(132)	(132)	(23)	(12)	(13)	I	(123)

with the property (3) of Remark 2, except that neither

$$\begin{matrix}(13)\\(132)\end{matrix} \quad \text{nor} \quad \begin{matrix}(132)\\(13)\end{matrix}$$

is in the subset; nevertheless $(S_3)^2 \subseteq D(\nu)^2$.

Table III — Partial table of M^2 for cross-connect corresponding to the permutation (13) (25) (46) and showing that condition of Remark 2 is satisfied

		M							
		1	2	3	4	5	6	7	8
M	1	I I							
	2	(23) I						(123) (12)	(13) (132)
	3	I (13)					(23) (123)	(13) (123)	
	4	(12) I				(132) (23)		(132) (12)	
	5	(13) (23)					(132) (132)	I (132)	(12) (12)
	6	(23) (12)				(123) (123)			(13) (13)
	7	(13) (12)				I (123)			
	8	(123) (132)						(23) (23)	

Table IV — Direct product elements corresponding to switch settings for cyclic cross-connect (5432)

sw #								
1st	11	21	11	21	11	21	21	11
	23	13	23	13	23	13	13	23
2nd	21	31	31	21	21	31	21	31
	32	22	22	32	32	22	32	22
3rd	12	32	12	32	32	12	12	32
	33	13	33	13	13	33	33	13
Elements of M	(12)	(132)	I	(132)	(23)	(13)	(12)	(13)
	(23)	(13)	(123)	(132)	(132)	(12)	(132)	(23)
	1	2	3	4	5	6	7	8

VI. CONJECTURE ABOUT NUMBER OF STAGES NEEDED TO GIVE REARRANGEABILITY WHEN A GIVEN CROSS-CONNECT FIELD IS USED

From Fig. 1 it is evident that an input switch on the left does not reach all the switches of the second stage, but can reach all the switches of the third stage hy passing through the second stage. Thus, regarding switches as vertices and links as edges, we can say that no input switch is farther away from a third-stage switch than $d = 2$ units, in the usual metric of the graph defined hy the vertices and edges. Furthermore, the number R of stages necessary and sufficient for rearrangeability is $5 = 2d + 1$. Similarly, in the three-stage network of Fig. 3, the distance from any input switch on the left to a middle

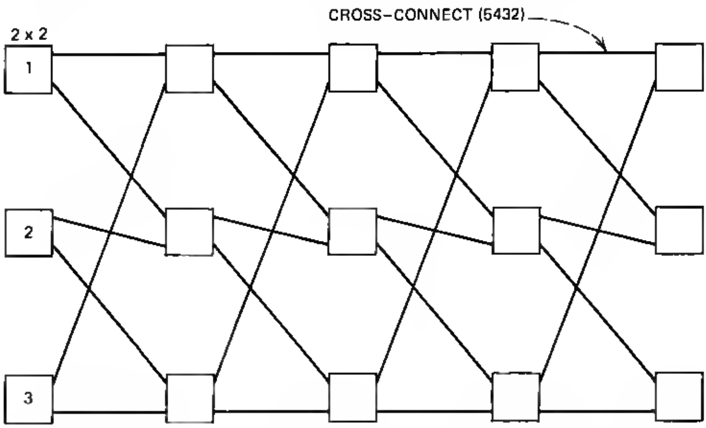


Fig. 6—Network based on cyclic cross-connect field corresponding to the permutation (5432).

Table V — Partial table of M^2 for cyclic cross-connect, showing that condition of Remark 2 is satisfied

M \ M	1	2	3	4	5	6	7	8
1	I I	(13) (123)				(132) (132)		
2	(23) (132)						(23) (23)	
3	(12) (12)		I (132)					
4	(23) (13)	(123) (12)	(132) (123)					
5	(132) (13)		(23) I					
6	(123) (123)							
7	I (13)	(13) (12)	(12) I					
8	(123) I	(23) (123)	(13) (13)	(23) (12)	(132) (12)			

switch is, of course, $d = 1$, and the number of stages R (necessary and sufficient for rearrangeability) is $3 = 2d + 1$. This leads us to suspect that there is a connection between the number of links one must go through to reach all switches of a stage and the number of stages needed to get a rearrangeable network.

To pose the question another way, let $\mathcal{S}$ be a stage of square switches, and φ a cross-connect field (permutation), and consider the natural sequence of networks such that

$$\begin{aligned} P(\nu_2) &= \mathcal{S}\varphi\mathcal{S} \\ P(\nu_3) &= \mathcal{S}\varphi\mathcal{S}\varphi\mathcal{S} \\ P(\nu_4) &= \mathcal{S}\varphi\mathcal{S}\varphi\mathcal{S}\varphi\mathcal{S} \\ &\vdots \end{aligned}$$

We ask for what value $s = R$ will ν_s first be rearrangeable, and how does this number R depend on φ?

Going back now to the graph defined by the switches as vertices and the links as edges, we shall say that an inlet switch or vertex has access to a switch in a given stage iff there is a path on the graph from

Table VI — Direct product elements corresponding to switch settings for cross-connect represented by (23) (45) (see Fig. 7)

sw #								
1st	11 22	12 21	11 22	11 22	12 21	11 22	12 21	12 21
2nd	11 33	11 33	13 31	11 33	13 31	13 31	11 33	13 31
3rd	22 33	22 33	22 33	23 32	23 32	23 32	23 32	22 33
Elements of M	I I	I (12)	I (13)	I (23)	(123) (132)	(23) (13)	(12) (23)	(12) (13)
	1	2	3	4	5	6	7	8

Table VII — Complete table of M^2 for cross-connect corresponding to the permutation (23) (45)

M	1	2	3	4	5	6	7	8
1	I* I	I* (12)	I* (13)	I* (23)	(123)* (132)	(23)* (13)	(12)* (23)	(12)* (13)
2	I (12)	I I	I* (132)	I* (123)	(123)* (13)	(23)* (132)	(12)* (123)	(12)* (132)
3	I (13)	I (123)	I I	I (132)	(123)* (23)	(23) I	(12)* (132)	(12) I
4	I (23)	I (132)	I (123)	I I	(123) (12)	(23) (123)	(12) I	(12) (123)
5	(123) (132)	(123) (23)	(123) (12)	(123) (13)	(132) (123)	(12)* (12)	(13)* (13)	(13) (12)
6	(23) (13)	(23) (123)	(23) I	(23) (132)	(13) (23)	I I	(132)* (132)	(132) I
7	(12) (23)	(12) (132)	(12) (123)	(12) I	(23) (12)	(123)* (123)	I I	I (123)
8	(12) (13)	(12) (123)	(12) I	(12) (132)	(23)* (23)	(123) I	I (132)	I I

Note: Neither $\genfrac{}{}{0pt}{}{(13)}{(132)}$ nor $\genfrac{}{}{0pt}{}{(132)}{(13)}$ occurs, so that condition of Remark 2 fails, although condition of Theorem 1 holds because

$$\genfrac{}{}{0pt}{}{\mathrm{I}}{(13)} \cdot \genfrac{}{}{0pt}{}{(13)}{(23)} = \genfrac{}{}{0pt}{}{(13)}{(132)}$$

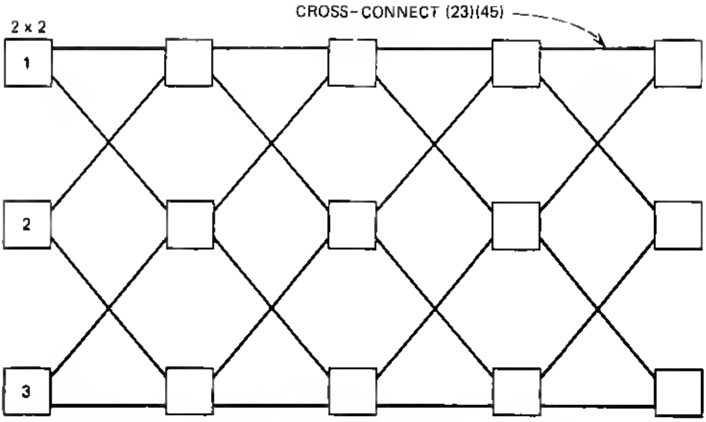


Fig. 7—Network based on cross-connect field corresponding to the permutation (23) (45).

the first switch to the second containing exactly one switch from every intermediate stage. Typically, the set of outlet switches to which an inlet switch has access will grow with the number of stages, and, for rearrangeability, it is, of course, necessary that every inlet switch have access to every outlet switch. Roughly speaking, the more access the field φ provides for switches from one stage to those of the next, the smaller will be the number of stages required for rearrangeability. It would therefore be of interest to relate this "amount of access" available with a given number of stages to the number of stages required for rearrangeability.

To this end, let us say that ν_s has "full access" if every inlet switch has access to every outlet switch, and define

$$d = \min \{s : \nu_{s+1} \text{ has full access}\}$$
$$R = \min \{s : \nu_s \text{ is rearrangeable}\}.$$

To return to the examples, if φ is the permutation (13) (25) (46) corresponding to the cross-connect field of Fig. 1, and $\mathcal{S}$ is a stage of three 2×2 switches, then $d = 2$ and $R = 5 = 2d + 1$. In Fig. 3, φ consists of a link between every pair of switches in successive stages, and so $d = 1$ and clearly $R = 3 = 2d + 1$. Again, in Fig. 6, using the cyclic cross-connect corresponding to (2345), it can be seen that $d = 2$ and $R = 5 = 2d + 1$.

All of these cases induce the following conjectures:

(*i*) ν_{2d+1} is rearrangeable.
(*ii*) $R = 2d + 1$.

It is easy to find additional confirming examples, especially necessity arguments for $R \geqq 2d + 1$, but to give a general proof seems to be very difficult.

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